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Convergence Rates for Multivariate Smoothing Spline Functions

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Let Ω be an open bounded subset of R^d, the d-dimensional space, and let f be an unknown function belonging to $H^m(\Omega)$, where m is an integer ($m > d/2$). Given the values of f at n scattered data point in Ω known with error, i.e., given $z_i = f(t_i) + \varepsilon_i$, $i = 1, ..., n$, where the ε_i's are i.i.d. random errors, we study the error $E[|f - \sigma_\lambda|^2_{k,\Omega}]$, where $|\cdot|^2_{k,\Omega}$ are the Sobolev semi-norms in $H^m(\Omega)$ and σ_λ is the thin plate smoothing spline with parameter λ, i.e., the unique minimizer of $\lambda |u|^2_m + (1/n) \sum_{i=1}^n (u(t_i) - z_i)^2$. Under the assumption that the boundary of Ω is smooth and the points satisfy a "quasi-uniform" condition, we obtain $E[|f - \sigma_\lambda|^2_{k,\Omega}] \leqslant C[\lambda^{(m-k)/m} |f|^2_{m,\Omega} + D/(n\lambda^{(2k+d)/2m})]$, $k = 0, 1, ..., m-1$.

1. INTRODUCTION

Let $H^m(\Omega)$, $m > 1$, be the Sobolev space

$$H^m(\Omega) = \left\{ u \in \mathscr{D}'(\Omega) \mid \int_\Omega \sum_{|\alpha| \leqslant m} |D^\alpha u|^2 < +\infty \right\}, \tag{1.1}$$

where $\mathscr{D}'(\Omega)$ is the space of Schwartz distributions and

$$D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} u \tag{1.2}$$

is the usual multi-index notation for partial derivatives.

For $m > d/2$, let f be an element of $H^m(\Omega)$ and let $z_1, z_2, ..., z_n$ satisfy

$$z_i = f(t_i) + \varepsilon_i, \qquad i = 1, ..., n, \tag{1.3}$$

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where $T = \{t_1, ..., t_n\}$ is a finite set of scattered data points in Ω and ε_i, $i = 1, ..., n$, are i.i.d. random variables with zero mean and variance v^2.

In order to approximate f it has been proposed to use the D^m-smoothing splines (cf. [8, 15, 18, 19]) defined as the unique solution of the variational problem

$$\underset{u \in D^{-m}L^2(\mathbb{R}^d)}{\text{Minimize}}\ J_\lambda(u), \tag{1.4}$$

where

$$J_\lambda(u) = \lambda \left[\sum_{i_1, ..., i_m = 1}^{d} \int_{\mathbb{R}^d} \left| \frac{\partial^m u(x)}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_m}} \right|^2 dx \right] + \frac{1}{n} \sum_{i=1}^{n} (u(t_i) - z_i)^2 \tag{1.5}$$

and

$$D^{-m}L^2(\mathbb{R}^d) = \{u \in \mathscr{D}'(\mathbb{R}^d) | D^\alpha u \in L^2(\mathbb{R}^d), \quad |\alpha| = m\}. \tag{1.6}$$

It has been proved by Duchon (cf. [9]) that (1.4) has a unique solution provided T contains a $\mathscr{P}_{m-1}$-unisolvent set, where $\mathscr{P}_{m-1}$ is the set of polynomials defined on R^d of total degree less than or equal to $m-1$. Moreover, Duchon [9] proved that the solution σ_λ is given by

$$\sigma_\lambda(t) = \sum_{i=1}^{n} c_i K_m(t - t_i) + p(t), \tag{1.7}$$

where K_m is the fundamental solution of the m-times iterated Laplacian and $p \in \mathscr{P}_{m-1}$, i.e.,

$$\Delta^m K_m = \delta. \tag{1.8}$$

Moreover, following Duchon [9] and Schwartz [14] we have for K_m

$$K_m(t) = \begin{cases} C_m |t|^{2m-d}, & d \text{ odd} \\ C_m |t|^{2m-d} \log |t|, & d \text{ even} \end{cases} \tag{1.9}$$

and

$$C_m = \begin{cases} \dfrac{(-1)^{d/2+1+m}}{2^{2m-1}\pi(d/2)(m-1)!(m-d/2)} & d \text{ even} \\ \dfrac{\Gamma(d/2-m)}{2^{2m}\pi(d/2)(m-1)!} & d \text{ odd} \end{cases}.$$

The coefficients $c_1, ..., c_n$ and the polynomial of degree $m-1$ can be found solving the following system of equations:

$$n\lambda c_i + \sum_{j=1}^{n} c_j K_m(t_i - t_j) + p(t_i) = z_i, \qquad i = 1, ..., n \tag{1.10}$$

$$\sum_{j=1}^{n} c_j q(t_j) = 0, \qquad \text{any } q \in m-1. \tag{1.11}$$

For details and computational procedures see [11, 12, 19].

$\lambda > 0$ is the smoothing parameter. For computational procedures to choose λ from the data see [7, 15, 19].

In this paper we aim to obtain error bounds of the type developed in [13] for the case $d=1$. The results obtained will be of the same type as those obtained by Cox [6] for the D^m-smoothing splines based on a bounded domain (cf. [3, 6, 15]).

More precisely, we will show the following result.

THEOREM 1.1. *Let Ω be an open bounded domain satisfying the uniform cone conditions and having a Lipschitz boundary. Define $h_{\max}$ and $h_{\min}$ as*

$$h_{\max} = \sup_{\tau \in \Omega} \inf_{i=1, ..., n} |t - t_i| \tag{1.12}$$

$$h_{\min} = \min_{i \neq j} |t_i - t_j| \tag{1.13}$$

and assume that there exists a constant $B > 0$ such that

$$\frac{h_{\max}}{h_{\min}} \leqslant B. \tag{1.14}$$

Then there exists $\lambda_0 > 0$ and constants P_0 and Q_0 such that

$$E[|f - \sigma_\lambda|_{j,\Omega}^2] \leqslant P_0 \lambda^{(m-j)/m} |f|_{m,\Omega}^2 + \frac{Q_0 v^2}{n\lambda^{(2j+d)/2m}} \tag{1.15}$$

for $\lambda \leqslant \lambda_0$ and $n\lambda^{d/2m} \geqslant 1$.

To prove this theorem we will first recall some basic properties of smoothing splines and give an expression for $E[|f - \sigma_\lambda|_{k,\Omega}^2]$ which will be used to bound the error. In the third section we show relationships between standard and discrete Sobolev norms. These results will be used in Section 4 to obtain the error estimates for exact data. In Section 5 we consider the study of the eigenvalues related to the spline problem and show how they behave asymptotically. The result generalizes a previous result of the author in one dimension and proves the conjecture by Wahba. In Section 6 we apply those results to bound the error in the presence of noise and prove the main theorem.

2. Basic Properties

In this section we collect several basic properties of D^m-smoothing splines that have been pointed out by several authors (cf. [8, 9, 6, 15]) and obtain the basic convergence rates.

Let us first introduce some notation. For $\mathbf{y} \in \mathbb{R}^n$ we call $S_{n,\lambda}(\mathbf{y})$ the smoothing spline of parameter λ applied to the data $y_1, ..., y_n$, i.e., the unique solution to

$$\underset{u \in D^{-m}L^2(\mathbb{R}^d)}{\text{Minimize}} \left\{ \lambda |u|_m^2 + \frac{1}{n} \sum_{i=1}^{n} (u(t_i) - y_i)^2 \right\}, \tag{2.1}$$

where

$$|u|_m^2 = \sum_{i_1, i_2, ..., i_m = 1}^{d} \int_{\mathbb{R}^d} \left| \frac{\partial^m u(x)}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_m}} \right|^2 dx. \tag{2.2}$$

Also for $g \in H^m(\Omega)$, $m > d/2$ we define $S_{n,\lambda}(g)$ as

$$S_{n,\lambda}(g) = S_{n,\lambda}(\mathbf{g}), \tag{2.3}$$

where

$$\mathbf{g} = (g(t_1), ..., g(t_n))^T \in \mathbb{R}^n. \tag{2.4}$$

For $u \in D^{-m}L^2(\mathbb{R}^d)$ we can define the bounded linear operator

$$T: D^{-m}L^2(\mathbb{R}^d) \to [L^2(\mathbb{R}^d)]^{d^m}$$

as

$$T(u) = \left(\frac{\partial^m u}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_m}} \right)_{i_1, i_2, ..., i_m = 1}^{d}. \tag{2.5}$$

Finally, in the space $[L^2(\mathbb{R}^d)]^{d^m} \times \mathbb{R}^n$ define the norm

$$||| [\tau, \mathbf{y}] |||^2 = \lambda \sum_{i_1, i_2, ..., i_m = 1}^{d} \int_{\mathbb{R}^d} \tau_{i_1, ..., i_d}^2 (x)\, dx + \frac{1}{n} \sum_{i=1}^{n} y_i^2. \tag{2.6}$$

With these notations $S_{n,\lambda}(\mathbf{y})$ is the unique solution to the problem

$$\underset{u \in D^{-m}L^2(\mathbb{R}^d)}{\text{Minimize}} ||| [Tu, \mathbf{u}] - [0, \mathbf{y}] |||^2. \tag{2.7}$$

Let $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ be the inner product in $[L^2(\mathbb{R}^d)]^{d^m} \times \mathbb{R}^n$ associated with $|\!|\!| \cdot |\!|\!|^2$. Using now (2.7) we easily obtain the following:

LEMMA 2.1. *For any* $d \in D^{-m}L^2(\mathbb{R}^d)$, $S_{n,\lambda}(\mathbf{y})$ *satisfies*

$$\lambda |g - S_{n,\lambda}(\mathbf{y})|_m^2 + \frac{1}{n} \sum_{i=1}^{n} (g(t_i) - S_{n,\lambda}(\mathbf{y})(t_i))^2 + \lambda |S_{n,\lambda}(\mathbf{y})|_m^2$$
$$+ \frac{1}{n} \sum_{i=1}^{n} (y_i - S_{n,\lambda}(\mathbf{y})(t_i))^2 = \lambda |g|_m^2 + \frac{1}{n} \sum_{i=1}^{n} (g(t_i) - y_i)^2. \qquad (2.8)$$

Proof. From (2.7) we observe that $[T(S_{n,\lambda}(\mathbf{y})), \mathbf{S}_{n,\lambda}(\mathbf{y})]$ is the projection of $[0, \mathbf{y}]$ onto

$$W = \{[Tu, \mathbf{u}] \text{ for } u \in D^{-m}L^2(\mathbb{R}^d)\}, \qquad (2.9)$$

hence from the orthogonality of the projection we get

$$|\!|\!| [T(g), \mathbf{g}] - [T(S_{n,\lambda}(\mathbf{y})), \mathbf{S}_{n,\lambda}(\mathbf{y})] |\!|\!|^2 + |\!|\!| [T(S_{n,\lambda}(\mathbf{y})), \mathbf{S}_{n,\lambda}(\mathbf{y})] - [0, \mathbf{y}] |\!|\!|^2$$
$$= |\!|\!| [T(g), \mathbf{g}] - [0, \mathbf{y}] |\!|\!|^2,$$

which is the same as (2.8). ∎

It is clear that $S_{n,\lambda}$ is a linear operator; then

$$S_{n,\lambda}(\mathbf{z}) = S_{n,\lambda}(f) + S_{n,\lambda}(\boldsymbol{\varepsilon}) \qquad (2.10)$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, ..., \varepsilon_n)^T$ is the vector containing the noise. This identity was also used by Ragozin [13] in the case of one-dimensional splines. Thus,

$$E[|S_{n,\lambda}(\mathbf{z}) - f|_{k,\Omega}^2] = |f - S_{n,\lambda}(f)|_{k,\Omega}^2 + E[|S_{n,\lambda}(\boldsymbol{\varepsilon})|_{k,\Omega}^2] \qquad (2.11)$$

since $E(\boldsymbol{\varepsilon}) = 0$. Here $|\cdot|_{k,\Omega}^2$ denotes the semi-norm

$$|u|_{k,\Omega}^2 = \sum_{i_1, ..., i_k = 1}^{d} \int_{\Omega} \left| \frac{\partial^k u(x)}{\partial x_{i_1} \cdots \partial x_{i_k}} \right|^2 dx. \qquad (2.12)$$

Also we have

$$E\left[\frac{1}{n} \sum_{i=1}^{n} |S_{n,\lambda}(\mathbf{z})(t_i) - f(t_i)|^2\right]$$
$$= \frac{1}{n} \sum_{i=1}^{n} [S_{n,\lambda}(f)(t_i) - f(t_i)]^2 + E\left[\frac{1}{n} \sum_{i=1}^{n} |S_{n,\lambda}(\boldsymbol{\varepsilon})(t_i)|^2\right]. \qquad (2.13)$$

In (2.11), (2.13) the first term corresponds to the error due to regularization of the exact function f while the second term is due only to the noise. To get an upper bound of the error due to the noise we will have to study the behavior of the eigenvalues of a matrix associated to the spline problem. This is done in Section 5. In this section we obtain the error bounds for the first term using the properties of $S_{n,\lambda}(f)$.

By substituting into (2.8) $g = f^{\Omega}$, the unique minimum semi-norm extension of f (see (3.2) below, we get

$$\lambda |f^{\Omega} - S_{n,\lambda}(f)|_m^2 + \frac{1}{n} \sum_{i=1}^{n} ((f(t_i) - S_{n,\lambda}(f)(t_i))^2 \leqslant \lambda |f^{\Omega}|_m^2,$$

thus

$$|f^{\Omega} - S_{n,\lambda}(f)|_m^2 \leqslant |f^{\Omega}|_m^2 \tag{2.14}$$

and

$$\frac{1}{n} \sum_{i=1}^{n} (f(t_i) - S_{n,\lambda}(f)(t_i))^2 \leqslant \lambda |f^{\Omega}|_m^2.$$

This concludes the proof of the following:

LEMMA 2.2. *We have the following error bounds*:

(i) $|f^{\Omega} - S_{n,\lambda}(f)|_m^2 \leqslant |f^{\Omega}|_m^2$;

(ii) $(1/n) \sum_{i=1}^{n} (f^{\Omega}(t_i) - S_{n,\lambda}(f)(t_i))^2 \leqslant \lambda |f^{\Omega}|_m^2$.

To use these results to obtain bounds for the terms $|f - S_{n,\lambda}(f)|_{k,\Omega}^2$ we need to relate $|g|_{0,\Omega}^2$, $|g|_{m,\Omega}$ and $(1/n) \sum_{i=1}^{n} [g(t_i)]^2$ and then use the interpolation theory of Sobolev spaces (cf. [1, 2]). We develop this relationship in the next section. In one dimension these results were first obtained by Ragozin [13]. In the case of multivariate splines Cox [6] and Wahba [18] have already given proofs of Lemma 2.2.

3. DISCRETE AND STANDARD SOBOLEV SEMI-NORMS

We begin this section recalling a result from Duchon [10].

LEMMA 3.1. *Let Ω be an open set of $\mathbb{R}^d$ satisfying a uniform cone condition, i.e., there exist $r > 0$ and $\theta > 0$ such that for any $t \in \Omega$ there exists a unit vector $\xi(t) \in \mathbb{R}^d$ such that the cone*

$$C(t, \xi(t), \theta, r) = \{t + \lambda\eta;\ \eta \in \mathbb{R}^d,\ |\eta| = 1,\ \eta \cdot \xi(t) \geqslant \cos\theta,\quad 0 \leqslant \lambda \leqslant r\}$$

is entirely contained in Ω.

Then there exist constants M_1 and M_2 (depending on d and θ) and ε (depending on θ and r) such that for any positive $\varepsilon \leqslant \varepsilon_0$, there exists $T_\varepsilon \subset \Omega$ satisfying

$$\text{(i)} \quad B(t, \varepsilon) \subset \Omega \text{ for any } t \in T_\varepsilon,$$

$$\text{(ii)} \quad \Omega \subset \bigcup_{t \in T_\varepsilon} B(t, M_1\varepsilon), \tag{3.1}$$

$$\text{(iii)} \quad \sum_{t \in T_\varepsilon} 1_{B(t, M_1\varepsilon)} \leqslant M_2,$$

where $B(t, R)$ is the closed ball of radius r centered t and 1_E is the function being equal to 1 for x in E and 0 for $x \notin E$.

Remark. Condition (iii) means that any point in Ω belongs to at most M_2 balls $B(t, M_1\varepsilon)$, $t \in T_\varepsilon$.

Proof. See [10].

For $g \in H^m(\Omega)$ we can define a unique extension g^Ω to $D^{-m}L^2(\mathbb{R}^d)$ by solving the variational problem (cf. [10])

$$\underset{\substack{u \in D^{-m}L^2(\mathbb{R}^d) \\ u|_\Omega = g}}{\text{Minimize}} \; |u|_m^2. \tag{3.2}$$

We have the following result connecting g and g^Ω.

LEMMA 3.2. *Let $r > 0$ be such that $B(0, R) \supset \Omega$. Then there exists a constant C (depending on R, m, d and Ω) such that for any $g \in H^m(\Omega)$*

$$|g^\Omega|_{0, B(0,R)}^2 + |g^\Omega|_{m, B(0,R)}^2 \leqslant C(|g|_{0,\Omega}^2 + |g|_{m,\Omega}^2). \tag{3.3}$$

Proof. In the space $D^{-m}L^2(\mathbb{R}^d)$ consider the norms

$$\|u\|_1^2 = |u|_{0,\Omega}^2 + |u|_m^2$$

$$\|u\|_2^2 = |u|_{0, B(0,R)}^2 + |u|_m^2.$$

Given that $D^{-m}L^2(\mathbb{R}^d)/\mathscr{P}_{m-1}$ is a Hilbert space, $D^{-m}L^2(\mathbb{R}^d)$ together with $\|\cdot\|_i$ is a Hilbert space for $i = 1, 2$. Moreover the injection j: $(D^{-m}L^2(\mathbb{R}^d), \|\cdot\|_2) \to (D^{-m}L^2(\mathbb{R}^d), \|\cdot\|_1)$ is continuous since

$$\|j(u)\|_1^2 = |u|_{0,\Omega}^2 + |u|_m^2 \leqslant \|u\|_2^2.$$

Using the open mapping theorem we obtain that j^{-1} is continuous, so there exists C_1 depending on R, Ω, d and m such that

$$\|u\|_2^2 = \|j^{-1}(u)\|_2^2 \leqslant C_1 \|u\|_1^2. \tag{3.4}$$

Thus for $u = g^{\Omega}$ we have

$$|g^{\Omega}|^2_{0,B(0,R)} + |g^{\Omega}|^2_{m,B(0,R)} \leqslant |g^{\Omega}|^2_{0,B(0,R)} + |g^{\Omega}|^2_m$$
$$\leqslant C_1(|g^{\Omega}|^2_{0,\Omega} + |g^{\Omega}|^2_{m,\Omega}).$$

Now we apply Lemma 3.1 in [10] to obtain that there exists K (depending on m, d and Ω) such that

$$|g^{\Omega}|^2_m \leqslant K|g|^2_{m,\Omega}. \tag{3.5}$$

We combine now (3.5) and (3.4) and get (3.3) for $C = \mathrm{Max}(C_1, C_1 K)$. ∎

We are now ready to show the first inequality connecting discrete and standard Sobolev norms. For the related result in the case $d = 1$ see Ragozin [13].

THEOREM 3.3. *Let $h_{\max}$ and $h_{\min}$ be defined as in* (1.12) *and* (1.13) *and let Ω be an open bounded set with Lipschitz boundary and satisfying a uniform cone condition. Then there exists a constant $B_0 > 0$ (depending only on m, d, Ω, B) and $h_0 > 0$ such that for any $g \in H^m(\Omega)$ $h = h_{\max} \leqslant h_0$ we have*

$$\frac{1}{n} \sum_{i=1}^{n} (g(t_i))^2 \leqslant B_0(|g|^2_{0,\Omega} + h^{2m}|g|^2_{m,\Omega}). \tag{3.6}$$

Proof. According to Lemma 3.1 there exists constants M_1, M_2 and $\varepsilon_0 > 0$ such that for $\varepsilon \leqslant \varepsilon_0$ there exists $T_\varepsilon \subset \Omega$ satisfying (3.1). Let $h_0 = M_1 \varepsilon_0$ then for $h = h_{\max} \leqslant h_0$ we can take $\varepsilon = h/M_1 \leqslant \varepsilon_0$ and obtain $T_h \subset \Omega$ such that

(i) $B(t, h/M_1) \subset \Omega$ for any $t \in T_\varepsilon$,

(ii) $\Omega \subset \bigcup_{t \in T_h} B(t, M_1\varepsilon) = \bigcup_{t \in T_h} B(t, h)$,

(iii) $\sum_{t \in T_h} 1_{B(t,h)} \leqslant M_2$.

Now, given (ii) for each i there exist $t \in T_h$ such that $t_i \in B(t, h)$. Consider now the classical technique of scaling and translation to the origin. Using the transformation $x \to (1/h)(x - t)$ the ball $B(t, h)$ is transformed into the ball $B(0, 1)$ and the point t_i into a point $\xi_i = (1/h)(t_i - t)$. Consider $g^{\Omega} \in D^{-m}L^2(\mathbb{R}^d)$ as defined by (3.2). It is clear that

$$\hat{g}^{\Omega}(\xi) = g^{\Omega}(t + h\xi)$$

also belongs to $D^{-m}L^2(\mathbb{R}^d)$, hence $\hat{g}^{\Omega}|_{B(0,2)} \in H^m(B(0, 2))$. But for any $v \in H^m(B(0, 2))$ we have $\forall \xi \in B(0, 1)$

$$|v(\xi)|^2 \leqslant \sup_{t \in B(0,1)} |v(t)|^2 \leqslant \mathrm{Const}[|v|^2_{0,B(0,2)} + |v|^2_{m,B(0,2)}]$$

since $m > d/2$. Then

$$[\hat{g}^{\Omega}(\xi_i)]^2 = [g^{\Omega}(t_i)]^2 = [g(t_i)]^2 \leqslant \text{Const}[|\hat{g}^{\Omega}|^2_{0,B(0,2)} + |\hat{g}^{\Omega}|^2_{m,B(0,2)}].$$

But

$$|\hat{g}^{\Omega}|^2_{0,B(0,2)} = h^{-d}|g^{\Omega}|^2_{0,B(t,2h)}$$

and

$$|\hat{g}^{\Omega}|^2_{m,B(0,2)} = h^{2m-d}|g^{\Omega}|^2_{m,B(t,2h)},$$

thus

$$[g(t_i)]^2 \leqslant \text{Const } h^{-d}[|g^{\Omega}|^2_{0,B(t,2h)} + h^{2m}|g^{\Omega}|^2_{m,B(t,2h)}].$$

In this way for each t_i we have selected one t such that $t_i \in B(t, h)$. Let us now add all the inequalities that we obtain in this way:

$$\sum_{i=1}^{n} [g(t_i)]^2 \leqslant \text{Const } h^{-d} \sum_{\substack{t(t_i)\\ i=1,\ldots,n}} \{|g^{\Omega}|^2_{B(t,2h)} + h^{2m}|g^{\Omega}|^2_{m,B(t,2h)}\}.$$

But it is clear that a particular t cannot be repeated in the sum more than $(2h_{\max}/h_{\min})^d$ times since that is an upper bound on the number of t_i's that $B(t, 2h)$ can contain, so

$$\sum_{i=1}^{n} [g(t_i)]^2 \leqslant \text{Const } h^{-d}\left[2\frac{h_{\max}}{h_{\min}}\right]^d \sum_{t \in T_h} \{|g^{\Omega}|^2_{0,B(t,2h)} + h^{2m}|g^{\Omega}|^2_{m,B(t,2h)}\}$$

$$h^d \sum_{i=1}^{n} (g(t_i))^2 \leqslant \text{Const } 2^d \left[\frac{h_{\max}}{h_{\min}}\right]^d M_2\{|g^{\Omega}|^2_{0,\Omega_h} + h^{2m}|g^{\Omega}|^2_{m,\Omega_h}\},$$

where $\Omega_h = \{x \in \mathbb{R}^d |\ |x-t| \leqslant 2h, \text{ any } t \in \Omega\}$.

Let $R > 0$ be such that $B(0, R) \supset \Omega_{h_0}$. Then for $h \leqslant h_0$, $\Omega_h \subset B(0, R)$ and we can apply Lemma 3.2 to obtain

$$h^d \sum_{i=1}^{n} [g(t_i)]^2 \geqslant \text{Const } 2^d \left[\frac{h_{\max}}{h_{\min}}\right]^d M_2\{|g^{\Omega}|^2_{0,B(0,R)} + h^{2m}|g^{\Omega}|^2_{m,B(0,R)}\}$$

$$\leqslant \text{Const } 2^d \left[\frac{h_{\max}}{h_{\min}}\right]^d M_2\{|g|^2_{0,\Omega} + h^{2m}|g|^2_{m,\Omega}\}. \tag{3.7}$$

Finally observe that

$$h^d \geqslant \frac{\text{Vol}(\Omega)}{2^d V_d}\frac{1}{n}, \tag{3.8}$$

where V_d is the volume of the unit sphere in $\mathbb{R}^d$.

Equations (3.7) and (3.8) give (3.6) for

$$B_0 = \frac{\text{Const } V_d 4^d}{\text{Vol}(\Omega)} M_2 C \left[\frac{h_{\max}}{h_{\min}}\right]^d.$$

This concludes the proof. ∎

Let $U = \{b_1, b_2, ..., b_M\}$, $M = \binom{d+m-1}{m-1} = \dim(\mathcal{P}_{m-1})$ be an arbitrary $\mathcal{P}_{m-1}$-unisolvent set in Ω. Let $\pi(U)$ be the Lagrange interpolation operator on $\mathcal{P}_{m-1}$, i.e., for any $v \in H^m(\Omega)$, $\pi(U)\, v$ is the unique element in $\mathcal{P}_{m-1}$ satisfying

$$(\pi(U)\, v)(b_i) = v(b_i), \qquad i = 1, 2, ..., M. \tag{3.9}$$

These polynomials have been extensively studied in the literature; see, for example, [4] and the references therein. Let us now prove the last result of this section.

THEOREM 3.4. *Under the same hypothesis of Theorem* 3.3 *there exists a constant* $C_0 > 0$ (*depending only on* m, d, Ω *and* B) *and* $h_0 > 0$ *such that for any* $g \in H^m(\Omega)$ *and* $h = h_{\max} \leqslant h_0$ *we have*

$$|g|_{0,\Omega}^2 \leqslant C_0 \left(\frac{1}{n} \sum_{i=1}^{n} [g(t_i)]^2 + h^{2m} |g|_{m,\Omega}^2\right). \tag{3.10}$$

Proof. Consider the $\mathcal{P}_{m-1}$-unisolvent set defined by

$$\hat{U} = \left\{\left(\frac{i_1}{m-1}, \frac{i_2}{m-1}, ..., \frac{i_d}{m-1}\right) \in \mathbb{R}^d: \; i_1, i_2, ..., i_d \text{ non-negative integers and } i_1 + i_2 + \cdots + i_d \leqslant m-1\right\}$$

(cf. [4]) and let $\hat{l}_i$, $i = 1, ..., M$, be the Lagrange polynomials associated with $\hat{U}$, i.e.,

$$\hat{l}_i(\hat{b}_j) = \begin{cases} 1, & i = j,\ \hat{b}_j \in \hat{U} \\ 0, & i \neq j \end{cases}. \tag{3.11}$$

Let $\hat{L}$ be defined as

$$\hat{L} = \max_{1 \leqslant i \leqslant M} \int_{B(0, 2M_1)} |\hat{l}_i(x)|^2 \, dx, \tag{3.12}$$

where M_1 is defined in Lemma 3.1. Now given that the set of $\mathcal{P}_{m-1}$-unisolvent sets is open in $\mathbb{R}^{Md}$ (its complement is the set of solutions of an algebraic equation) there exists $0<\delta<1$ such that if we choose b_i, $i=1, ..., M$, satisfying

$$|b_i-\hat{b}_i|\leqslant\delta \tag{3.13}$$

the set $\mathscr{B}=\{b_1, ..., b_M\}$ is $\mathscr{P}_{m-1}$-unisolvent. Also the Lagrange polynomials l_i^b, $i=1, ..., M$, associated with $\mathscr{B}$ depend continuously on $\{b_1, ..., b_M\}$, so δ may be chosen so that

$$L^b=\operatorname*{Max}_{1\leqslant i\leqslant M}\int_{B(0,2M_1)}|l_i^b(x)|^2\,dx\leqslant 2\hat{L} \tag{3.14}$$

holds $\forall b$ satisfying (3.13).

Let us now apply Lemma 3.1 for $\varepsilon=2h/\delta$ and $h\leqslant h_0=2\delta\varepsilon_0$. For any $t\in T_\varepsilon$ consider the transformation

$$\xi\to t+\frac{h}{\delta}\xi,$$

which transforms $B(0,2)$ into $B(t,\varepsilon)\subset\Omega$. Also the balls $B(\hat{b}_i,\delta)$ are transformed into $B(t+(h/\delta)\,\hat{b}_i,h)$. Thus for $i=1, ..., M$ there exists a point $t_{r(i)}\in T_\varepsilon$ such that $t_{r(i)}\in B(t+(h/\delta)\,\hat{b}_j,h)$. Moreover, given the definition of δ, the points $\{t_{r(i)}\}_{i=1}^M$ form a $\mathscr{P}_{m-1}$-unisolvent set. Let us denote by U^Ω the set $\{t_{r(i)}\}_{i=1}^M$ and by $U=\{b_i=(\delta/h),\ (t_{r(i)}-t),\ i=1, ..., M\}\subset B(0,2)$. We then have

$$\int_{B(t,M_1\varepsilon)}|\pi(U^\Omega)\,g(x)|^2\,dx=\left(\frac{h}{\delta}\right)^d\int_{B(0,2M_1)}|\pi(U)\,\tilde{g}(\xi)|^2\,d\xi$$

for $\tilde{g}(\xi)=g(t+(h/\delta)\,\xi)$. Then

$$\begin{aligned}\int_{B(t,M_1\varepsilon)}|\pi(U^\Omega)\,g(x)|^2\,dx&=\left(\frac{h}{\delta}\right)^d\int_{B(0,2M_1)}\left|\sum_{i=1}^M l_i^b(\xi)\,\tilde{g}(b_i)\right|^2 d\xi\\&\leqslant\left(\frac{h}{\delta}\right)^d M\sum_{i=1}^M[\tilde{g}(b_i)]^2\int_{B(0,2M_1)}|l_i^b(\xi)|^2\,d\xi\\&\leqslant h^d\frac{2M}{\delta^d}\hat{L}\sum_{i=1}^M(\tilde{g}(b_i))^2\\&\leqslant h^d\frac{2M\hat{L}}{\delta^d}\sum_{i=1}^M(g(t_{r(i)}))^2.\end{aligned}$$

On the other hand we can apply the Lemma of Section 2 in [10] to conclude that there exists a constant D_0 depending only on Ω, m, d and δ such that

$$\begin{aligned}|g^\Omega-\pi(U^\Omega)\,g^\Omega|^2_{0,B(t,M_1\varepsilon)}&=|g^\Omega-\pi(U^\Omega)\,g|^2_{0,B(t,M_1\varepsilon)}\\&\leqslant D_0h^{2m}|g^\Omega|^2_{m,B(t,M_1\varepsilon)}.\end{aligned}$$

Thus

$$|g^{\Omega}|^2_{0,B(t,M_1\varepsilon)} \leqslant 2[|\pi(U^{\Omega})\,g|^2_{0,B(t,M_1\varepsilon)} + D_0 h^{2m}|g^{\Omega}|^2_{m,B(t,M_1\varepsilon)}].$$

Now we use the fact that $\bigcup_{t\in T_\varepsilon} B(t, M_1\varepsilon) \supset \Omega$ and conclude that

$$|g^{\Omega}|^2_{0,\Omega} \leqslant 2\sum_{t\in T_\varepsilon}\left(\frac{2M}{\delta^d}\hat{L}\right) h^d \sum_{i=1}^{M} (g(t_{r(i)}))^2 + 2D_0 h^{2m} \sum_{t\in T_\varepsilon} |g^{\Omega}|^2_{m,B(t,M_1\varepsilon)},$$

and using now (3.1)(iii) we obtain

$$|g^{\Omega}|^2_{0,\Omega} \leqslant 2M_2\left(\frac{2M\hat{L}}{\delta^d}\right) h^d \sum_{i=1}^{n} (g(t_i))^2 + 2M_2\, D_0 h^{2m}|g^{\Omega}|^2_m.$$

We then apply Lemma 3.1 in [10] to conclude that

$$|g^{\Omega}|^2_{0,\Omega} \leqslant 2M_2\left(\frac{2M\hat{L}}{\delta^d}\right) h^d \sum_{i=1}^{n} (g(t_i))^2 + (2M_2\, KD_0)\, h^{2m}|g|^2_{m,\Omega}.$$

Finally we use the fact that

$$h^d \leqslant \frac{1}{n}\frac{\mathrm{Vol}(\Omega)}{V_d}\left(\frac{h_{\max}}{h_{\min}}\right)^d$$

and obtain (3.10) with

$$C_0 = \mathrm{Max}\left\{\frac{4M_2 M\hat{L}}{\delta^d}\frac{\mathrm{Vol}(\Omega)}{V_d}\left(\frac{h_{\max}}{h_{\min}}\right)^d, 2M_2\, KD_0\right\}. \tag{3.15}$$

This concludes the proof. ∎

In the next section we use these results to obtain an error bound for $|f - S_{n,\lambda}(f)|^2_{j,\Omega}$, $j = 0, ..., m$.

4. Error Bounds for Exact Data

We can now use Theorem 3.4 for $g = f^{\Omega} - S_{n,\lambda}(f)$, Lemma 2.2 and (3.5) to obtain

$$\begin{aligned} |f - S_{n,\lambda}(f)|^2_{0,\Omega} &\leqslant C_0\left[\frac{1}{n}\sum_{i=1}^{n} (f(t_i) - S_{n,\lambda}(f)(t_i))^2 + h^{2m}|f - S_{n,\lambda}(f)|^2_{m,\Omega}\right] \\ &\leqslant C_0\,[\lambda K|f|^2_{m,\Omega} + h^{2m}K|f|^2_{m,\Omega}] \\ &\leqslant \lambda\, C_0 K\left[1 + \frac{1}{\lambda h^{-2m}}\right]|f|^2_{m,\Omega}. \end{aligned} \tag{4.1}$$

Also

$$|f-S_{n,\lambda}(f)|^2_{m,\Omega} \leqslant |f^{\Omega}-S_{n,\lambda}(f)|^2_m \leqslant K|f|^2_{m,\Omega}. \tag{4.2}$$

To get bounds for the intermediate derivatives we only need to apply the interpolation inequality (see, for example, [1, Theorem 4.14]), which gives

$$|f-S_{n,\lambda}(f)|^2_{j,\Omega} \leqslant P(\theta^{-(2j/(m-j)}|f-S_{n,\lambda}(f)|^2_{0,\Omega} + \theta^2|f-S_{n,\lambda}(f)|^2_{m,\Omega})$$

for each $\theta \leqslant \theta_0$.

Let us take $\theta = \lambda^{(m-j)/2m}$. Then for $\lambda^{(m-j)/2m} \leqslant \theta_0$, $j=1, ..., m-1$, or equivalently $\lambda \leqslant \theta_0^{2m}$ $(\theta_0 < 1)$, we have

$$\begin{aligned}|f-S_{n,\lambda}(f)|^2_{j,\Omega} &\leqslant P(\lambda^{-j/m}|f-S_{n,\lambda}(f)|^2_{0,\Omega} + \lambda^{(m-j)/m}|f-S_{n,\lambda}(f)|^2_{m,\Omega}) \\ &\leqslant P\left(\lambda^{-j/m}\lambda C_0 K\left[1+\frac{1}{\lambda h^{-2m}}\right]|f|^2_{m,\Omega} + \lambda^{(m-j)/m} K|f|^2_{m,\Omega}\right) \\ &\leqslant P\,K\left[1+C_0+\frac{C_0}{\lambda h^{-2m}}\right]\lambda^{(m-j)/m}|f|^2_{m,\Omega}. \end{aligned} \tag{4.3}$$

From here it is easy to get the following result.

THEOREM 4.1. *Let $f \in H^m(\Omega)$, where Ω is an open bounded set with Lipschitz boundary and satisfying a uniform cone condition. Let $h = h_{\max}$ and $h_{\min}$ be defined as in* (1.12), (1.13) *and assume that there exists a constant $B > 0$ such that*

$$\frac{h_{\max}}{h_{\min}} \leqslant B. \tag{4.4}$$

Then there exist $\lambda_0 > 0$ and $P_0 > 0$ such that for any $0 < \lambda \leqslant \lambda_0$ and $n\lambda^{d/2m} \geqslant 1$ we have

$$|f-S_{n,\lambda}(f)|^2_{j,\Omega} \leqslant P_0 \lambda^{(m-j)/m}|f|^2_{m,\Omega}, \qquad 0 \leqslant j \leqslant m. \tag{4.5}$$

Proof. Given the definition of $h_{\min}$ and M_1 (cf. Theorem 3.3) It is clear that

(i) $B(t_i, h_{\min}/M_1) \subset \Omega$;

(ii) $B(t_j, h_{\min}/M_1) \cap B(t_i, h_{\min}/M_1) = \varnothing$.

Thus

$$\begin{aligned}\text{Vol}(\Omega) &\geqslant \sum_{i=1}^{n} \text{Vol}\left(B\left(t_i, \frac{h_{\min}}{M_1}\right)\right) \\ &\geqslant n\left(\frac{h_{\min}}{M_1}\right)^d V_d \\ &\geqslant n\left(\frac{h}{M_1}\right)^d V_d\left(\frac{h_{\min}}{h_{\max}}\right)^d \end{aligned}$$

or

$$\left(\frac{h}{M_1}\right)^d \leqslant \frac{1}{n}\frac{\mathrm{Vol}(\Omega)}{V_d\cdot}\left[\frac{h_{\max}}{h_{\min}}\right]^d$$

$$h^{2m} \leqslant M_1^{2m}\frac{1}{n^{2m/d}}\left[\frac{(\mathrm{Vol}(\Omega)}{V_d}\right]^{2m/d} B^{2m}.$$

Hence

$$\frac{h^{2m}}{\lambda} \leqslant (M_1 B)^{2m}\left[\frac{\mathrm{Vol}(\Omega)}{V_d}\right]^{2m/d}\frac{1}{n^{2m/d}\lambda}$$

and using the hypothesis we get

$$\frac{h^{2m}}{\lambda} \leqslant (BM_1)^{2m}\left[\frac{\mathrm{Vol}(\Omega)}{V_d}\right]^{2m/d}.$$

We finally put

$$P_0 = \mathrm{Max}\left\{1, P\left[1 + C_0 + C_0(BM_1)^{2m}\left[\frac{\mathrm{Vol}(\Omega)}{V_d}\right]^{2m/d}\right]\right\} \tag{4.6}$$

and use (4.3) to get the desired result.

5. Noisy Data and Eigenvalues Associated to the Thin Plate Splines

In Section 2 we developed the basic formulas to bound the error $f - \sigma_\lambda$. The bounds obtained contain two terms, one the error due to the smoothing of exact data as studied in the preceding section and the other that due strictly to the noise. More precisely (2.11) gives

$$E[|f-\sigma_\lambda|_{k,\Omega}^2] = |f - S_{n,\lambda}(f)|_{k,\Omega}^2 + E[|S_{n,\lambda}(\varepsilon)|_{k,\Omega}^2]$$

and (2.13)

$$E\left[\frac{1}{n}\sum_{i=1}^{n}|\sigma_\lambda(t_i) - f(t_i)|^2\right]$$
$$= \frac{1}{n}\sum_{i=1}^{n}[f(t_i) - S_{n,\lambda}(f)(t_i)]^2 + E\left[\frac{1}{n}\sum_{i=1}^{n}(S_{n,\lambda}(\varepsilon)(t_i))^2\right].$$

The first term of these two expressions can be bounded using the results of the last two sections. To bound the last term we first write it in a standard form (cf. [6, 15, 18]).

First define the energy matrix Γ as the one representing the quadratic form

$$\mathbf{y}^T \Gamma \mathbf{y} = \underset{\substack{u \in D^{-m}L^2(\mathbb{R}^d) \\ u(t_i) = y_i\ i=1,\ldots,n}}{\text{Minimum}} |u|_m^2. \tag{5.1}$$

The solution $\hat{u}$ to (5.1) is the thin plate spline interpolating the data $y_1, \ldots, y_n$ at the knots of T. Using this matrix we can now write (cf. [15, 19]) (1.5) as follows:

$$\underset{u \in D^{-m}L^2(\mathbb{R}^d)}{\text{Min}} \left\{ \lambda |u|_m^2 + \frac{1}{n} \sum_{i=1}^{n} (u(t_i) - w_i)^2 \right\}$$
$$= \underset{\mathbf{x} \in \mathbb{R}^n}{\text{Min}} \left\{ \lambda \mathbf{x}^T \Gamma \mathbf{x} + \frac{1}{n} \sum_{i=1}^{n} (x_i - w_i)^2 \right\}. \tag{5.2}$$

Thus the solution $\hat{x} = (S_{n,\lambda}(\mathbf{w})(t_1), \ldots, S_{n,\lambda}(\mathbf{w})(t_n))^T$ is given by

$$\hat{x} = (I + n\lambda\Gamma)^{-1}\, \mathbf{w} = A(\lambda)\, \mathbf{w}. \tag{5.3}$$

And we can write for $\mathbf{w} = \boldsymbol{\varepsilon}$

$$\frac{1}{n} \sum_{i=1}^{n} (S_{n,\lambda}(\boldsymbol{\varepsilon}))(t_i))^2 = \frac{1}{n} \sum_{i=1}^{n} (\hat{x}_i)^2$$
$$= \frac{1}{n}\, \boldsymbol{\varepsilon}^T A^2(\lambda)\, \boldsymbol{\varepsilon},$$

which together with the properties of ε_i, $i = 1, \ldots, n$, gives

$$E\left[\frac{1}{n} \sum_{i=1}^{n} (S_{n,\lambda}(\boldsymbol{\varepsilon}))(t_i))^2\right] = \frac{1}{n}\, v^2\, \mathrm{Tr}(A^2(\lambda)). \tag{5.4}$$

On the other hand

$$|S_{n,\lambda}(\boldsymbol{\varepsilon})|_m^2 = \hat{x}^T \Gamma \hat{x}$$
$$= \boldsymbol{\varepsilon}^T A(\lambda)\, \Gamma A(\lambda)\, \boldsymbol{\varepsilon}$$
$$= \frac{1}{n\lambda}\, \boldsymbol{\varepsilon}^T [A(\lambda)(A^{-1}(\lambda) - I)\, A(\lambda)]\, \boldsymbol{\varepsilon}$$
$$= \frac{1}{n\lambda}\, \boldsymbol{\varepsilon}^T [A(\lambda) - A^2(\lambda)]\, \boldsymbol{\varepsilon}.$$

Also,

$$E[|S_{n,\lambda}(\varepsilon)|_m^2] = \frac{1}{n\lambda} v^2[\operatorname{Tr}(A(\lambda)) - \operatorname{Tr}(A^2(h))]$$

$$\leqslant \frac{1}{n\lambda} v^2 \operatorname{Tr}(A(\lambda)). \tag{5.5}$$

We can see that in the study of error bounds for the right-hand sides of (5.4) and (5.5) the behavior of $\operatorname{Tr}(A(\lambda))$ and $\operatorname{Tr}(A^2(\lambda))$ is of fundamental importance. We will analyze these quantities using the eigenvalues of $A(\lambda)$ or, more precisely those of $n\Gamma$. Let $0 \leqslant \mu_{1n} \leqslant \mu_{2n} \leqslant \cdots \leqslant \mu_{nn}$ be the eigenvalues of $n\Gamma$ in ascending order. Clearly they are all non-negative real numbers since Γ is non-negative definite, and obviously

$$\operatorname{Tr}(A(\lambda)) = \sum_{i=1}^{n} \frac{1}{1+\lambda\mu_{in}} \tag{5.6}$$

$$\operatorname{Tr}(A^2(\lambda)) = \sum_{i=1}^{n} \frac{1}{(1+\lambda\mu_{in})^2}. \tag{5.7}$$

We now proceed to study the eigenvalues of $n\Gamma$ using a similar technique to that developed in [17] for the one-dimensional case.

THEOREM 5.1. *Let Ω be an open bounded domain with Lipschitz boundary and satisfying a uniform cone condition and let $\{t_1, ..., t_n\}$ satisfy (1.14). Then there exist constants $\alpha, \beta > 0$ (depending on Ω, $h_{\max}/h_{\min}$ and m) such that*

$$\alpha\mu_i \leqslant \mu_{in} \leqslant \beta\mu_i, \qquad i = 1, 2, ..., n, \tag{5.8}$$

where $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$ are the first n eigenvalues of the variational eigenvalue problem

$$(\psi, \phi)_m = \mu \int_\Omega \psi\phi \qquad \text{for any } \phi \in D^{-m}L^2(\mathbb{R}^d) \tag{5.9}$$

and

$$(\psi, \phi)_m = \int_{R^d} \sum_{i_1, i_2, ..., i_m = 1}^{d} \frac{\partial^m \psi}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_m}} \cdot \frac{\partial^m \phi}{\partial x_{i_1} \partial x_{i_2} \cdots \partial xi_m} \, dx \tag{5.10}$$

is the semi-inner product associated with $|\cdot|_m^2$.

Proof. (a) For $i = 1, ..., M$, $\mu_{in} = \mu_i = 0$ and (5.8) holds.

It is clear that (5.9) holds for any $\psi \in \mathscr{P}_{m-1}$, $\mu = 0$ and $\phi \in D^{-m}L^2(\mathbb{R}^d)$, thus $\mu_1 = \mu_2 \cdots \mu_M = 0$ since $M = \binom{d+m-1}{m\ \ 1}$ dimension of $\mathscr{P}_{m-1}$. But

$\mu_{M+1} \neq 0$ since $(\psi, \phi)_m = 0$ any $\phi \in D^{-m}L^2(\mathbb{R}^d)$ implies that $|\psi|_m^2 = 0$ and $\psi \in \mathscr{P}_{m-1}$.

On the other hand let $y_i = \psi(t_i)$, $i = 1, ..., n$, for $\psi \in \mathscr{P}_{m-1}$; then clearly

$$\mathbf{y}^T \Gamma \mathbf{y} = 0$$

since the thin plate spline is the unique function in $D^{-m}L^2(\mathbb{R}^d)$ taking the values at y_i at t_i minimizing $|\cdot|_m^2$ and $|\psi|_m^2 = 0$.

Since Γ is symmetric this implies that $\mu_{1n} = \mu_{2n} = \cdots \mu_{Mn} = 0$.

On the other hand if $\mu_{M+1,n} = 0$, the $M + 1$ eigenvector is formed by the values of a polynomial of degree $m - 1$, which contradicts the fact that the algebraic and geometric multiplicities must coincide. Thus $\mu_{M+1,n} > 0$.

(b) Let $\mathbf{x}$ be an eigenvector of $n\Gamma$ corresponding to a non-zero eigenvalue. We have

$$n\Gamma \mathbf{x} = \mu \mathbf{x}$$

$$\Gamma \mathbf{x} = \mu \frac{1}{n} \mathbf{x}$$

or

$$\mathbf{y}^T \Gamma \mathbf{x} = \mu \frac{1}{n} \mathbf{y}^T \mathbf{x} \qquad \text{any } \mathbf{y} \in \mathbb{R}^n. \tag{5.11}$$

Now, let $\phi \in D^{-m}L^2(\mathbb{R}^d)$ be such that

$$\phi(t_i) = x_i, \qquad i = 1, 2, ..., n,$$

and let s be the thin plate spline interpolating $y_1, ..., y_n$ at $t_1, ..., t_n$. Then clearly

$$\mathbf{y}^T \Gamma \mathbf{x} = (s, \phi)_m \tag{5.12}$$

and (5.11) becomes

$$(s, \phi)_m = \mu \frac{1}{n} \sum_{i=1}^{n} s(t_i)\, \phi(t_i) \qquad \text{any } \phi \in D^{-m}L^2(\mathbb{R}^d). \tag{5.13}$$

Thus any eigenvalue of $n\Gamma$ is also an eigenvalue of (5.13) and vice versa. Moreover, consider the eigenvalue problem:

Find $\psi \in D^{-m}L^2(\mathbb{R}^d)$ such that

$$(\psi, \phi)_m = \mu \frac{1}{n} \sum_{i=1}^{n} \psi(t_i)\, \phi(t_i) \qquad \text{any } \phi \in D^{-m}L^2(\mathbb{R}^d). \tag{5.14}$$

This problem, which is a "discretized" version of (5.9) has only n finite eigenvalues. The reason for this becomes clear when we consider the Raleigh quotient formulation of (5.14) (cf. [5, 20]). But let us first observe that any eigenvalue μ of (5.14) is an eigenvalue of (5.13). To see this we only have to prove that the corresponding eigenfunction is a thin plate spline, which is clear from (5.14) since it implies that

$$\Delta^m \psi = \mu \frac{1}{n} \sum_{i=1}^{n} \psi(t_i)\, \delta_{t_i}, \tag{5.15}$$

which is the characterization of a thin plate spline (cf. [8]).

Thus, μ_{in} is characterized by the min–max formulation

$$\mu_{in} = \underset{\substack{\operatorname{codim}(V) = i-1 \\ V \text{ subspace of } D^{-m}L^2(\mathbb{R}^d)}}{\operatorname{Max}} \ \underset{u \in V}{\operatorname{Min}} \frac{|u|_m^2}{(1/n)\sum_{j=1}^{n} [u(t_j)]^2}. \tag{5.16}$$

For $i = n+1$, the subspace

$$V = \left\{ \phi \in D^{-m}L^2(\mathbb{R}^d) \Big/ \sum_{j=1}^{n} \phi(t_j)\, \psi_{k,n}(t_j) = 0, \quad k = 1, ..., n \right\},$$

where $\psi_{1n}, ..., \psi_{nn}$ are the first n eigenfunctions of (5.14), has codimension $i - 1 = n$.

Moreover, for $\phi \in V$ we have $|\phi|_m^2 \neq 0$ ($\phi \notin \mathscr{P}_{m-1}$) and $\phi(t_j) = 0, j = 1, ..., n$, since $\{\psi_{k,n}\}_{k=1}^{n}$ is a basis for the space of thin plate splines. (It is a set of n orthogonal elements of the space of dimension n.) Thus, for $\phi \in V$ we have

$$\frac{|\phi|_m^2}{(1/n)\sum_{i=1}^{n} (\phi(t_i))^2} = +\infty \tag{5.17}$$

and

$$\mu_{n+1,n} = +\infty. \tag{5.18}$$

The same reasoning works for $i \geqslant n+1$ and thus the eigenvalues of the matrix problem are given by (5.16) only for $i = 1, ..., n$.

Now we can use Theorem 3.3 and get for any $\phi \in D^{-m}L^2(\mathbb{R}^d)$ with $\sum_{i=1}^{n} [\phi(t_j)]^2 \neq 0$

$$\frac{|\phi|_m^2}{(1/n)\sum_{j=1}^{n}[\phi(t_j)]^2} \geqslant \frac{|\phi|_m^2}{B_0\{|\phi|_{0,\Omega}^2 + h^{2m}|\phi|_m^2\}}. \tag{5.19}$$

Thus

$$\mu_{in} \geqslant \frac{1}{B_0}\, \rho_{in}, \qquad i = 1, 2, ..., n, \tag{5.20}$$

where $\rho_{1n} \leqslant \cdots \leqslant \rho_{nn}$ are the first n eigenvalues of the variational eigenvalue problem

$$(\psi, \phi)_m = \rho[(\psi, \phi)_{0,\Omega} + h^{2m}(\psi, \phi)_m] \qquad \text{any } \phi \in D^{-m}L^2(\mathbb{R}^d),$$

which are given by

$$\rho_i = \frac{\mu_i}{1 + h^{2m}\mu_i}, \qquad i = 1, 2, ..., n, \tag{5.21}$$

Thus (5.20) implies

$$\mu_{in} \geqslant \frac{1}{1 + h^{2m}\mu_i} \frac{1}{B_0} \mu_i, \qquad i = 1, ..., n.$$

But as we shall see later $\mu_i h^{2m}$, $i = 1, ..., n$, is bounded from above, i.e, there exists $\alpha > 0$ such that

$$\alpha \leqslant \frac{1}{B_0(1 + h^{2m}\mu_i)}, \qquad i = 1, 2, ..., n, \tag{5.22}$$

which gives the first inequality in (5.8).

Using now Theorem 3.4 we get

$$\frac{|\phi|_m^2}{|\phi|_{0,\Omega}^2} \geqslant \frac{|\phi|_m^2}{C_0[(1/n) \sum_{j=1}^n [\phi(t_j)]^2 + h^{2m}|\phi|_m^2]},$$

which implies that

$$\mu_i \geqslant \frac{1}{C_0} \xi_{in}, \qquad i = 1, 2, ..., n, \tag{5.23}$$

where the ξ_{in}'s are the first n eigenvalues of

$$(\psi, \phi)_m = \xi \left[\frac{1}{n} \sum_{i=1}^{n} \psi(t_j)\, \phi(t_j) + h^{2m}(\psi, \phi)_m \right] \qquad \text{any } \phi \in D^{-m}L^2(\mathbb{R}^d),$$

which are given by

$$\xi_{in} = \frac{\mu_{in}}{1 + h^{2m}\mu_{in}}, \qquad i = 1, 2, ..., n. \tag{5.24}$$

Thus

$$\begin{aligned} \mu_{in} &\leqslant C_0 \mu_i (1 + h^{2m}\mu_{in}) \\ &\leqslant C_0 \mu_i (1 + h^{2m}\mu_{nn}). \end{aligned} \tag{5.25}$$

(c) $h^{2m}\mu_{nn}$ is bounded. To see this, let ϕ_{nn} be the corresponding eigenfunction. Then, given that ϕ_{nn} is a thin plate spline, we have

$$|\phi_{nn}|_m^2 = \underset{\substack{u \in D^{-m}L^2(\mathbb{R}^d) \\ u(t_i) = \phi_{nn}(t_i) \quad i=1,\dots,n}}{\mathrm{Min}} |u|_m^2. \tag{5.26}$$

Let us now define ω as the $C^\infty(\mathbb{R}^d)$ function with support $B(0, 1)$:

$$\omega(s) = \begin{cases} e^{-|s|^2/(1-|s|^2)}, & 0 \leqslant |s| \leqslant 1 \\ 0, & |s| > 1. \end{cases} \tag{5.27}$$

Thus, $\omega \in D^{-m}L^2(\mathbb{R}^d)$ and so does

$$\omega_i(t) = \omega\left(\frac{t - t_i}{h_{\min}}\right). \tag{5.28}$$

Moreover, given the definition of $h_{\min}$, we have

$$\omega_i(t_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

Thus

$$u = \sum_{j=1}^{n} \phi_{nn}(t_j)\, \omega_j \tag{5.29}$$

belongs to $D^{-m}L^2(\mathbb{R}^d)$ and $u(t_j) = \phi_{nn}(t_j)$, $j = 1, \dots, n$. Using (5.26) we get

$$|\phi_{nn}|_m^2 \leqslant |u|_m^2.$$

But given the definition u and the ω_j's we have

$$|\phi_{nn}|_m^2 \leqslant |u|_m^2 = \sum_{j=1}^{n} [\phi_{nn}(t_j)]^2 |\omega_j|_m^2. \tag{5.30}$$

To compute $|\omega_j|_m^2$ we use (5.28) and conclude that

$$|\omega_j|_m^2 = h_{\min}^{-2m+d} |\omega|_m^2. \tag{5.31}$$

This together with (5.30) finally gives

$$|\phi_{nn}|_m^2 \leqslant h_{\min}^{-2m} \left\{ h_{\min}^d \sum_{j=1}^{n} [\phi_{nn}(t_j)]^2 \right\} |\omega|_m^2.$$

And using (1.14) again

$$h_{\min}^d \leqslant h^d \leqslant \frac{M_1^d \operatorname{Vol}(\Omega)}{n V_d}$$

and

$$h_{\min}^{-2m} \leqslant B^{2m} h^{-2m}$$
$$|\phi_{nn}|_m^2 \leqslant h^{-2m} \frac{B^{2m} M_1^d \operatorname{Vol}(\Omega)}{V_d} |\omega|_m^2 \frac{1}{n} \sum_{j=1}^{n} [\phi_{nn}(t_j)]^2,$$

which implies that

$$\mu_{nn} \leqslant h^{-2m} \frac{B^{2m} M_1^d \operatorname{Vol}(\Omega)}{V_d},$$

proving that $h^{2m}\mu_{nn}$ is bounded. This together with (5.25), except for the problem with (5.22), concludes the proof of the theorem. ▌

Now we study the behavior of the eigenvalues $\mu_1 \leqslant \mu_2 \leqslant \cdots$.

LEMMA 5.2. *Let Ω be an open bounded domain with Lipschitz boundary and satisfying a uniform cone condition. Let $\mu_1 \leqslant \mu_2 \leqslant \cdots$ be the eigenvalues of the variational problem* (5.9) *and let $\rho_1 \leqslant \rho_2 \leqslant \cdots$ be the eigenvalues of*

$$(\psi, \phi)_{m,\Omega} = \mu \int_\Omega \psi\phi \qquad \textit{for any } \phi \in H^m(\Omega). \tag{5.32}$$

Then there exists a constant $K(\Omega)$ such that

$$\rho_i \leqslant \mu_i \leqslant K(\Omega)\, \rho_i, \qquad i = 1, 2, \ldots. \tag{5.33}$$

Proof. Let ϕ_i be the eigenfunction of (5.9) corresponding to the eigenvalue μ_i normalized by $|\phi_i|_{0,\Omega}^2 = 1$, and let ψ_i be the eigenfunction of (5.26) corresponding to the eigenvalue ρ_i normalized by $|\psi_i|_{0,\Omega}^2 = 1$.

As has been proved in [10] under the given hypothesis on Ω there exists a unique ϕ_i^Ω extending ϕ_i to $D^{-m}L^2(\mathbb{R}^d)$ which minimizes $|\cdot|_m^2$. On the other hand $\phi_i|_\Omega$, the restriction of ϕ_i to Ω, belongs to $H^m(\Omega)$, hence it can be extended uniquely to $D^{-m}L^2(\mathbb{R}^d)$ with minimum $|\cdot|_m^2$ seminorm. Let ϕ_i^Ω be that extension. We first prove that $\phi_i^\Omega = \phi_i$. We have

$$\mu_i = |\phi_i|_m^2 = \operatorname*{Min}_{\substack{(u,\phi_j)_{0,\Omega} = 0 \\ 1 \leqslant j \leqslant i-1 \\ |u|_{0,\Omega}^2 = 1}} |u|_m^2,$$

but $(\phi_i^{\Omega}, \phi_j)_{0,\Omega} = 0$ since $\phi_i^{\Omega} = \phi_i$ on Ω and $|\phi_i^{\Omega}|_m^2 \leqslant |\phi_i|_m^2$; then

$$\mu_i = \operatorname*{Min}_{\substack{(u,\phi_j)_{0,\Omega}=0 \\ 1 \leqslant j \leqslant i-1 \\ |u|_{0,\Omega}^2 = 1}} |u|_m^2 \leqslant |\phi_i^{\Omega}|_m^2 \leqslant |\phi_i|_m^2$$

and we conclude that

$$|\phi_i|_m^2 = |\phi_i^{\Omega}|_m^2.$$

The extension being unique we conclude that $\phi_i = \phi_i^{\Omega}$. Thus μ_i, ϕ_i can also be defined by

$$\mu_i = \operatorname*{Min}_{\substack{(u,\phi_j)_{0,\Omega}=0 \\ 1 \leqslant j \leqslant i-1 \\ |u|_{0,\Omega}^2 = 1 \\ u \in H^m(\Omega)}} |u^{\Omega}|_m^2. \tag{5.34}$$

But according to Duchon [10] there exists a constant $K(\Omega)$ depending only on Ω such that for any $u \in H^m(\Omega)$ we have

$$|u^{\Omega}|_m^2 \leqslant K(\Omega)|u|_{m,\Omega}^2,$$

hence

$$\mu_i = \operatorname*{Min}_{\substack{(u,\phi_j)_{0,\Omega}=0 \\ 1 \leqslant j \leqslant i-1 \\ u \in H^m(\Omega) \\ |u|_{0,\Omega}^2 = 1}} |u^{\Omega}|_m^2 \leqslant K(\Omega) \operatorname*{Min}_{\substack{(u,\phi_j)_{0,\Omega}=0 \\ j = 1,\ldots,i-1 \\ u \in H^m(\Omega) \\ |u|_{0,\Omega}^2 = 1}} |u|_{m,\Omega}^2$$

$$\leqslant K(\Omega) \operatorname*{Max}_{\substack{V \text{ subspace of } H^m(\Omega) \\ \operatorname{Codim}(V) = i-1}} \operatorname*{Min}_{\substack{u \in V \\ |u|_{0,\Omega}^2 = 1}} |u|_m^2 = K(\Omega)\,\rho_i.$$

On the other hand for any $u \in H^m(\Omega)$

$$|u|_{m,\Omega}^2 \leqslant |u^{\Omega}|_m^2,$$

thus

$$\rho_i = \operatorname*{Min}_{\substack{(u,\psi_j)_{0,\Omega}=0 \\ 1 \leqslant j \leqslant i-1 \\ u \in H^m(\Omega) \\ |u|_{0,\Omega}^2 = 1}} |u|_{0,\Omega}^2 \leqslant \operatorname*{Min}_{\substack{(u,\psi_j)_{0,\Omega}=0 \\ 1 \leqslant j \leqslant i-1 \\ u \in H^m(\Omega) \\ |u|_{0,\Omega}^2 = 1}} |u^{\Omega}|_m^2$$

but for any $v \in D^{-m}L^2(\mathbb{R}^d)$ we have $|v^{\Omega}|_m^2 \leqslant |v|_m^2$ and $\{r \in H^m(\Omega) | r = v|_{\Omega}$ some $v \in D^{-m}L^2(\mathbb{R}^d)\} = H^m(\Omega)$, hence

$$\operatorname*{Min}_{\substack{(u,\psi_j)_{0,\Omega}=0\\1\leqslant j\leqslant i-1\\u\in H^m(\Omega)\\|u|^2_{0,\Omega}=1}} |u^\Omega|^2_m = \operatorname*{Min}_{\substack{(u,\psi_j)_{0,\Omega}=0\\1\leqslant j\leqslant i-1\\u\in D^{-m}L^2(\mathbb{R}^d)\\|u|^2_{0,\Omega}=1}} |u^\Omega|^2_m$$

$$\leqslant \operatorname*{Max}_{\substack{V\text{ subspace of }D^{-m}L^2(\mathbb{R}^d)\\\operatorname{codim}(V)=i-1}} \operatorname*{Min}_{\substack{u\in V\\|u^\Omega|^2_{0,\Omega}=1}} |u^\Omega|^2_m = \mu_i.$$

This concludes the proof. ∎

We are now ready to give the main result of this section.

THEOREM 5.3. *Let Ω be an open, bounded subset of $\mathbb{R}^d$ with Lipschitz boundary and satisfying a uniform cone condition and let $\mu_{1n} \leqslant \mu_{2n} \cdots \leqslant \mu_{nn}$ be the eigenvalues of $n\Gamma$. Then there exist constants $\alpha_1, \beta_1 > 0$ such that for $M+1 \leqslant i \leqslant n$ we have*

$$i^{2m/d}\alpha_1 \leqslant \mu_{in} \leqslant \beta_1 i^{2m/d}. \tag{5.35}$$

Proof. According to Theorem 5.1 and Lemma 5.2 it suffices to prove that the eigenvalues $\rho_1 \leqslant \rho_2 \leqslant \cdots$ satisfy a relationship of the type (5.35). To see this we observe that $\rho_1 \leqslant \rho_2 \leqslant \cdots$ are the eigenvalues of the differential operator $(-1)^m \Delta^m$ which is unbounded in $L^2(\Omega)$ but symmetric in $C^\infty(\Omega)$ with appropriate boundary conditions. We can then apply Theorem 14.6 in [2] to conclude that the number of eigenvalues of this operator less than or equal to $C_+^{-2m/d} i^{2m/d}$ is $i(1+o(1))$. C_+ is a constant independent of i where $o(1)$ goes to zero as i increases. Thus there exist integers N_1, N_2 such that the number of ρ_j's less than or equal to $C_+^{-2m/d} i^{2m/d}$ is between $N_1 i$ and $N_2 i$ for any i. Thus

$$\rho_1 \leqslant \rho_2 \cdots \leqslant \rho_{N_1 i} \leqslant C_+^{-2m/d} i^{2m/d} \tag{5.36}$$

and

$$\rho_{N_2(i+1)} \geqslant C_+^{-2m/d} i^{2m/d}. \tag{5.37}$$

Thus

$$(N_2 C_+)^{-2m/d}(i-1)^{2m/d} \leqslant \rho_i \leqslant (N_i C_+)^{-2m/d}(i)^{2m/d}, \qquad i = M+1, \ldots.$$

This concludes the proof. ∎

This result had been conjectured by Wahba [18] in the general case and has been proved by the author in one dimension in the case of equally spaced data and general m in [17] and in the case of arbitrary spaced data for cubic splines in [16]. Related work in several dimensions has been done by Cox [6].

Remark. From this result and (5.34) it is clear that

$$\mu_i \leqslant K(\Omega)(N_1 C_+)^{-2m/d} i^{2m/d}.$$

But we also have $h^{2m} = O(n^{-2m/d})$ and conclude that

$$h^{2m}\mu_i \leqslant K(\Omega)(N_1 C_+)^{-2n/d} C' \left(\frac{1}{n}\right)^{2m/d}$$

is bounded.

6. Error Bounds for Noisy Data

With the results of the preceding section we can conclude the proof of the main result of this paper.

Proof of Theorem 1.1. From (5.4) we have

$$\begin{aligned}
E\left[\frac{1}{n}\sum_{i=1}^{n}(S_{n,\lambda}(\varepsilon))\, t_i))^2\right] &= \frac{1}{n} v^2 \operatorname{Tr}(A^2(\lambda)) \\
&= \frac{1}{n} v^2 \sum_{i=1}^{n} \frac{1}{(1+\lambda\mu_{in})^2} \\
&\leqslant \frac{1}{n} v^2 \left[M + \sum_{j=M+1}^{n} \frac{1}{(1+\lambda C j^{2m/d})^2}\right] \\
&\leqslant \frac{1}{n} v^2 \left[M + \lambda^{-d/2m} \int_M^\infty \frac{1}{(1+Cx^{2m/d})^2}\, dx\right] \\
&\leqslant \frac{v^2}{n\lambda^{d/2m}} \left[M\lambda^{d/2m} + \int_M^\infty \frac{1}{(1+Cx^{2m/d})^2}\, dx\right].
\end{aligned}$$

For some constant C, thus if we set

$$Q_1 = M\lambda_0^{d/2m} + \int_M^\infty \frac{1}{(1+Cx^{2m/d})^2}\, dx \tag{6.1}$$

we have

$$E\left[\frac{1}{n}\sum_{i=1}^{n}(S_{n,\lambda}(\varepsilon))(t_i)^2\right] \leqslant \frac{v^2 Q_1}{n\lambda^{d/2m}} \qquad \text{for } \lambda \leqslant \lambda_0. \tag{6.2}$$

On the other hand from (5.5)

$$\begin{aligned}E[|S_{n,\lambda}(\varepsilon)|_m^2] &\leqslant \frac{1}{n\lambda} v^2 \operatorname{Tr}(A(\lambda)) \\ &\leqslant \frac{1}{n\lambda} v^2 \left[M + \sum_{j=M+1}^{n} \frac{1}{1+\lambda C j^{2m/d}} \right] \\ &\leqslant \frac{1}{n\lambda} v^2 \left[M + \lambda^{-d/2m} \int_M^\infty \frac{1}{1+Cx^{2m/d}}\, dx \right].\end{aligned}$$

Thus if we set

$$Q_2 = M\lambda_0^{d/2m} + \int_M^\infty \frac{1}{1+Cx^{2m/d}}\, dx \tag{6.3}$$

we have

$$E[|S_{n,\lambda}(\varepsilon)|_m^2] \leqslant \frac{Q_2 v^2}{n\lambda\lambda^{d/2m}}, \tag{6.4}$$

where the integrals in (6.2) and (6.4) converge since $m > d/2$.

We can now use Theorem (3.4) and get

$$\begin{aligned}E[|S_{n,\lambda}(\varepsilon)|_{0,\Omega}^2] &\leqslant C_0 \left\{ E\left[\frac{1}{n} \sum_{i=1}^{n} [S_{n,\lambda}(\varepsilon)(t_i)]^2 + h^{2m} E[|S_{n,\lambda}(\varepsilon)|_m^2 \right]\right\} \\ &\leqslant C_0 v^2 \left[\frac{1}{n\lambda^{d/2m}} + \frac{Q_2 h^{2m}}{n\lambda\lambda^{d/2m}} \right] \\ &= \frac{C_0 v^2}{n\lambda^{d/2m}} \left[Q_1 + \frac{Q_2}{h^{-2m}\lambda} \right].\end{aligned}$$

But $N\lambda^{d/2m} \geqslant 1$; then if we set

$$Q_3 = Q_1 + Q_2 (B\, M_1)^{2m} \left[\frac{\operatorname{Vol}(\Omega)}{V_d} \right]^{2m/d} \tag{6.5}$$

we have

$$E[|S_{n,\lambda}(\varepsilon)|_{0,\Omega}^2] \leqslant \frac{Q_3 C_0 v^2}{n\lambda^{d/2m}}. \tag{6.6}$$

Finally we use the interpolation theorem for indermediate derivatives (cf., for example, Theorem 4.14 in [1]) to get

$$E[|S_{n,\lambda}(\varepsilon)|_{k,\Omega}^2] \leqslant P(\lambda^{-k/m} E[|S_{n,\lambda}(\varepsilon)|_{0,\Omega}^2] + \lambda^{(m-k)/m} E[|S_{n,\lambda}(\varepsilon)|_{m,\Omega}^2])$$
$$\leqslant P\left(\lambda^{-k/m} \frac{Q_3 C_0 v^2}{n\lambda^{d/2m}} + \lambda^{(m-k)/m} \frac{Q_2 v^2}{n\lambda\lambda^{d/2m}}\right)$$
$$\leqslant \frac{P(Q_3 C_0 + Q_2)\, v^2}{n\lambda^{(2k+d)/2m}}$$
$$\leqslant \frac{Q_0 v^2}{n\lambda^{(2k+d)/2m}} \tag{6.7}$$

for $Q_0 = P(Q_3 C_0 + Q_2)$.

Finally we combine (2.11), (4.5) and (6.7) to obtain the desired result. ▮

A similar result was obtained by Cox [6] for the case of splines defined on a finite domain Ω. Also for $d = 1$ this result was first proved by Ragozin [13] in the case of equally spaced data points. Finally, let us remark that the results of this theorem were already conjectured by Wahba [18].

To conclude this section we state the following straightforward corollary of Theorem 1.1.

COROLLARY 6.1. *Let Ω be an open bounded domain of $\mathbb{R}^d$ with Lipschitz boundary and satisfying a uniform cone condition. Let $f \in H^m(\Omega)$, $m > d/2$, and $\{t_1, ..., t_n\}$ contain at least a $\mathscr{P}_{m-1}$-unisolvent set and*

$$\frac{h_{\max}}{h_{\min}} \leqslant B.$$

Then the optimum upper bound on the rate of convergence is attained for

$$\lambda^* = O(n^{-2m/(2m+d)})$$

and is given by

$$E[|f - \sigma_{n,\lambda^*}|_{k,\Omega}^2] = O(n^{-2(m+k)/(2m+d)}). \tag{6.8}$$

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